Filter Banks For "Intensity Analysis"

Introduction
In connection with our participation in two conferences in Canada (August 2002) we met von Tscharner several times discussing his "intensity analysis" method for EMG. Basically the method is a sort of bandpassing the signal. The connection with wavelet theory is that the filter is constructed by rescaling a given mother wavelet using a special array of scales (center frequencies – see equ (2) below) with non-constant relative bandwidth. A closer theoretical analysis was undertaken in order to evaluate the content of the proposed method.

1. The von Tscharner filter bank
The "intensity analysis" method proposed by von Tscharner (2000) is based on a set of wavelets (known as the Cauchy/Paul/Poisson wavelets) defined in frequency space by (restricted to \( f \geq 0 \) by the Heaviside function \( \Theta(f) \); that is, it is a progressive/prograde wavelet)

\[
\hat{\psi}(f_c, \text{scale}, f) = \left( \frac{f}{f_c} \right)^{\eta} e^{\left( 1 - \frac{f}{f_c} \right)^{\eta}} \cdot \Theta(f)
\]

with
\[ \eta = \text{scale} \cdot f_c \]

for an array of center frequencies given by

\[
f_c = \frac{1}{\text{scale}} (q + j)^r
\]

for \( j = 0, 1, ..., J \). Fixed parameter values chosen by von Tscharner are\(^1\)

\[
q = 1.45 \\
r = 1.959 \\
\text{scale} = 0.3
\]

With these values the summation of the eleven \((J = 10)\) first wavelets in frequency space gives a function that is almost constant in the interval of 20 to 200 Hz. The wavelets can thus be used as a filter bank for decomposing signals into frequency bands in this interval (for larger \( J \) we get a bigger interval). That the sum of (1) is nearly constant \((C)\) in this interval,

\[
\sum_j \hat{\psi}_j(f) \approx C
\]

seems to imply that the corresponding sum in time-space

\(^1\) It would be more convenient to put \( r \) exactly equal to 2. This would ensure that the sum of filters would on the average be constant and not have an increasing trend as is the case with the choice \( r = 1.959 \). See further § 3.

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approximates the Dirac delta (the inverse Fourier transform of the constant unit function). However, due to the restriction to the non-negative frequencies \( f \geq 0 \) we have instead\(^2\)

\[
\sum_j \psi_j(t) \approx C \left( \frac{\delta(t)}{2} + \frac{i}{2\pi t} \right)
\]

(here we have written \( \hat{\psi}_j(f) \) for \( \hat{\psi}(f_{cj}, \text{scale}, f) \) etc). In this sense a signal can be approximately decomposed into a sum, according to

\[
x_j(t) = \frac{1}{C} \int \hat{\psi}_j(u-t)x(u)du
\]

\[
x(t) \approx 2 \sum \Re(x_j(t))
\]

Thus, the real part of \( x_j(t) \) gives a description how the "component" of \( x(t) \) centered on the frequency \( fc_j \) behaves with time. In the "intensity analysis" proposed by von Tscharner one is rather interested in tracking the "intensity" for the various components,

\[
p_j(t) = |x_j(t)|^2
\]

If we sum (6) over \( j \) and integrate over time we get (via Parseval's relation)

\[
\sum_j \int p_j(t)dt = \int \sum_j |\hat{\psi}_j(f)|^2 |\hat{x}(f)|^2 df
\]

and this is proportional to the "energy" \( \int |\hat{x}(f)|^2 df \) of the signal if we may assume that

\[
\sum_j |\hat{\psi}_j(f)|^2
\]

too is almost constant in the frequency interval. With these points in mind (6) may be interpreted as a measure of the strength of the signal around time \( t \) and frequency \( fc_j \).

In physical space the wavelet (1) becomes a complex function

\(^2\) The rhs is (in distributional sense) for the case when the frequency interval becomes infinite.
\[
\psi(f_c, \eta, t) = \Gamma(\eta+1) e^{\eta} \frac{f_c}{(\eta - i 2 \pi f_c t)^{\eta+1}}
\]

with
\[
\eta = \text{scale} \cdot f_c
\]

One can check that the norm of the wavelet is given by (using the Plancherel theorem)

\[
\int_{-\infty}^{\infty} |\psi(f_c, \eta, t)|^2 dt = e^{2\eta} \frac{\Gamma(2\eta+1)}{(2\eta)^{2\eta+1}} f_c
\]

The frequency resolution is calculated to be

\[
\Delta f = f_c \cdot \frac{1}{2\eta} \left(1 + \frac{1}{2\eta}\right)
\]

and the mean frequency

\[
\langle f \rangle = f_c \left(1 + \frac{1}{2\eta}\right)
\]

whereas (1) attains the maximum value 1 at \( f = f_c \).

An interesting consequence of equ (10) is that it solves the following integral for any non-negative real number \( \eta \),

\[
\int_{0}^{\infty} \frac{dt}{(1+t^2)^{\eta+1}} = \frac{\Gamma(2\eta+1)}{\Gamma(\eta+1)} 2^{-2\eta-1}
\]

Using (13) we may compute the time resolution to be (only defined for \( \eta > 0.5 \))

\[
\Delta t = \frac{1}{2\pi f_c} \frac{\eta}{\sqrt{2\eta-1}}
\]
Thus, the time-frequency "uncertainty" relation becomes

\[
\Delta f \cdot \Delta t = \frac{1}{4\pi} \sqrt{\frac{2\eta + 1}{2\eta - 1}}
\]

which is very close to the optimal result \(1/4\pi\) when \(\eta >> 1\).

A sinus-wave of frequency \(f_0\) whose representation in frequency space is

\[
\frac{1}{2i} \left( \delta(f - f_0) - \delta(f + f_0) \right)
\]

will be transformed by the wavelet \((1)\) to (in physical space)

\[
\frac{1}{2i} \left( \frac{f_0}{f_c} \right)^\eta \cdot e^{(1 - \frac{f}{f_c})\eta + i2\pi f_0 t}
\]

If the sinus-wave is restricted to a length \(T\) we have to replace \((16)\) by

\[
\frac{1}{2i} \left( \frac{\sin(\pi (f - f_0) T)}{\pi (f - f_0)} - \frac{\sin(\pi (f + f_0) T)}{\pi (f + f_0)} \right)
\]

Another choice of a pulselike signal might be the Gaussian pulse

\[
e^{-\frac{t^2}{2\Delta^2}} \sin(2\pi f_0 t)
\]

whose Fourier-transform is

\[
\sqrt{2\pi \cdot \Delta t} \cdot \left( e^{-2\pi^2 \Delta^2 t (f-f_0)^2} - e^{-2\pi^2 \Delta^2 t (f+f_0)^2} \right)
\]

In the case of the input signal is the sinus wave of frequency \(f_0\) and length \(T\) ((whose Fourier-transform is \((18)\)) the convolution \(g(f_c, f_0, \eta, T, t)\) with the (complex conjugated) analyzing wavelet
(9) can be conveniently calculated from the integral

\begin{equation}
(21) \quad g(f_c, f_0, \eta, T, t) = \frac{1}{2} \int_0^T \left( \bar{\psi} \left( f_c, \eta, t - \frac{u}{2} \right) - \bar{\psi} \left( f_c, \eta, t + \frac{u}{2} \right) \right) \sin(\pi f_0 u) du
\end{equation}

The figure below shows the output in case of $T = 70$ ms, $f_0 = 60$ Hz and analyzing wavelet $j = 3$.

![Output Diagram](image1)

The output is an imaginary function, as is true in the case (17) too (corresponding to $T \to \infty$), and we have displayed the real and imaginary parts in fig. 1. The magnitude of the output is shown in fig. 2.

![Magnitude Diagram](image2)
The real and imaginary parts are apparently phase-shifted by 90 degrees relative each other (as in (17)). Von Tscharner (2000, p. 438) describes a somewhat intricate method for calculating the "intensity" for the transformed signal \( v \). If \( v_j \) is the transform of the signal by the wavelet \( j \) then the "intensity" \( p_j \) is calculated as

\[
p_j(t) = v_j(t)^2 + \left( \frac{1}{2\pi f_c} \frac{d v_j(t)}{dt} \right)^2
\]

The last term on the RHS in (22) is included according to von Tscharner in order to get rid of the oscillatory terms when the transform is e.g. applied to a sinus-wave. However, this extra term is not needed if we use both the real and imaginary parts of the wavelet (9) and we take the magnitude of the complex transformed function as a measure of the intensity. As we see from fig. 1 the real and imaginary parts by themselves oscillate, but the magnitude of the complex transformed function has a nice bell-shape (fig. 2). The method (22) will give the same result in case of an infinite sinus-wave but in other cases only approximately so.

The fig. 3 below shows the norms

\[
\int_{-\infty}^{\infty} |g(t)|^2 dt
\]

of the corresponding outputs for analyzing wavelets \( j = 0, \ldots, J \) applied to our finite sinus-wave.

Though peaked near \( f_0 \) this distribution has a quite pronounced tail. If we calculate the standard
variation $\Delta f$ of the frequency distribution we get a value around 6.4 Hz which implies that $\Delta f \times T \approx 0.45$. The case with an infinite sinus wave as the input signal is shown in the next figure.

![Graph showing variation in frequency distribution](image)

**Fig. 4**

2. A Morlet filter bank

One can obtain a similar filter bank using Morlet "wavelets"\(^3\)

\[
\hat{\psi}(f_c, \alpha, f) = e^{\frac{-n^2}{2\alpha f_c} (f-f_c)^2}
\]

The summation of these wavelets is shown below using $\alpha = 150$ and the center frequencies (1.2) and $J = 10$.

\(^3\) Compared to the standard form of the Morlet the parameter $\alpha$ is here multiplied with $f_c$. See further § 3.

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The time and frequency resolutions are given by

\[ \Delta t = \frac{1}{\sqrt{2 \alpha f_c}} \]  
\[ \Delta f = \frac{\sqrt{2 \alpha f_c}}{4\pi} \]

and are thus optimal in term of the time-frequency relation. These relations agree approximately with (1.11) and (1.14) if choose \( \alpha \) as

\[ \alpha = \frac{4\pi^2}{\text{scale}} \quad \text{when} \quad \eta = \text{scale} \cdot f_c. \]

A few of the corresponding Morlet wavelets in physical space are shown below given by

\[ \psi(t) = \sqrt{\alpha f_c} e^{-i2\pi f t - \alpha f_c t^2/2} \]

which is Gaussian unlike (1.9). In the present case the time resolution varies from 22 ms to 3 ms and the frequency resolution from 3.6 Hz to 36.4 Hz \((j = 0, \ldots, 10)\). From (3) it follows that the frequency resolution grows linearly with \( j \) if \( r \) is close to 2 in (1.2).
Strictly speaking the Morlet wavelets are not proper wavelets because they do not satisfy the admissibility condition, still (1) is very small for $f = 0$ so this circumstance may have not much practical implications anyway. The Morlet wavelets and Paul wavelets give within the approximations used here practically identical results, but the Morlet has perhaps some nicer mathematical properties from the computational point of view.

3. Discussion

The new idea with von Tscharner's method seems to be the use the special set (1.2) of center frequencies. Also the analysis is not primarily based on an attempt to decompose the signal into a sum of wavelets but to use a filter bank strictly for power analysis. The "standard" Morlet wavelet in frequency space is written (non-normalized)

$$\hat{\psi}(\alpha, f_c, f) = e^{-\frac{(f-f_c)^2}{\alpha^2}}$$

Note that here the $f_c$-factor is dropped after $\alpha$ in the exponent as compared to (2.1). Thus, the formulas corresponding to (2.2-3) become in this case

$$\Delta t = \frac{1}{\sqrt{2\alpha}}$$

$$\Delta f = \frac{\sqrt{2\alpha}}{4\pi}$$

The continuous wavelet transform is usually calculated for scales and frequencies parametrized as
This will imply, using (3), that the relative bandwidth \( BW \) defined by

\[
BW = \frac{\Delta f}{f_c}
\]

remains constant. The previous filter banks differ by having a non-constant relative band-width. The von Tscharner filter bank has a relative bandwidth which decreases as

\[
BW \approx \frac{1}{\sqrt{\text{scale} \cdot f_c}}
\]

when \( \text{scale} \cdot f_c \gg 1 \). Indeed, in order that the sum

\[
\sum_j \hat{\psi}(\alpha_j, f_j, f)
\]

be approximately independent of \( f \) in the range of interest, the frequency resolution \( \Delta f \) should scale approximately as \( f_{j+1} - f_j \); that is, given (1.2) (with \( r = 2 \)) we should have

\[
\Delta f \sim \sqrt{f}
\]

This is indeed satisfied if we choose the form (2.1) for the scaling of the Morlet function. We can illustrate the situation employing a much simpler division of the frequency band by using rectangles (see fig. below).
Thus, suppose we have boxes centered at $f_j$ and with widths $\Delta_j$ related by,

\begin{equation}
  f_{j+1} - f_j = \Delta_{j+1} + \Delta_j
\end{equation}

From this it follows that if $f_j$ is a quadratic function of the index $j$,

\begin{equation}
  f_j = a(q + j)^2
\end{equation}

then we get for the (half-) box sizes

\begin{equation}
  \Delta_j = a(q + j) = \sqrt{af_j}
\end{equation}

which can be compared with (5). This indicates a general relationship between center-frequencies (7) and widths for filter banks that try to minimize the overlap. Naturally, the box-filter is far from ideal when we consider its behaviour in time (the "ringing" sinc-function)\(^4\),

\begin{equation}
  \psi_j(t) = e^{i2\pi ft} \left( \frac{\sin(2\pi \Delta_j t)}{\pi t} \right)
\end{equation}

The signal $x$ is analyzed by the wavelets (2.1) as

\(^4\) If we try to determine the time resulution $\Delta t$ for (9) from the second time moment we find that it diverges.
(10) \[ c_j(t) = \int \hat{\psi}_j(u-t)x(u)du \]

where \( \psi_j \) is shorthand for \( \psi(\alpha_j, f, t) \). In frequency space we get

(11) \[ \sum |\hat{c}_j(f)|^2 = \sum |\hat{\psi}_j(f)|^2 |\hat{x}(f)|^2 \]

Thus, if the sum \( \sum |\hat{\psi}_j(f)|^2 \) is also approximately constant in the frequency band then (11) will be proportional to the power of the signal at the frequency \( f \). The value \( |c_j(t)|^2 \) may thus be taken as a measure of the power of the signal around frequency \( f_j \) at the time \( t \). So in this sense the condition (1-8) of the constancy of the squared amplitudes of the wavelet in frequency space is required, whereas the condition (1-3) is only indirectly related to this. Anyway, this condition requires that adjacent wavelets \( \psi_j \) and \( \psi_{j+1} \) overlap enough so that the wavelets cover the frequency band without gaps. Furthermore, in order that the "intensity" \( p_j \) may be interpreted as measure of the energy in a frequency band centered on \( f_{c,j} \) the wavelets \( \psi_j \) and \( \psi_{j+2} \) must have a minimal overlap and thus separate the different subbands (minimal "leakage" between bands). By using a relative bandwidth of the form (4) these objectives can be approximately fulfilled. Von Tscharner (2000) provides thus an interesting but a somewhat ad hoc recipe.

The use of non-normalized wavelets (1.1) and (2.1) brings in a further aspect related to the concept of equalizer (EQ) familiar from audio-technology. This refers to the ability to independently amplify the signal in different frequency bands. The conventional continuous wavelet transform (CWT) for a scaling parameter \( a \) is defined by

(12) \[ T(a, t) = \frac{1}{\sqrt{a}} \int \bar{\psi} \left( \frac{u-t}{a} \right) x(u) du \]

using a normalization factor \( 1/\sqrt{a} \). In frequency space (12) becomes

(13) \[ \hat{T}(a, f) = \sqrt{a} \bar{\psi}(a f) \hat{x}(f) \]

The scalings (4) correspond to using \( a = p^{-j} \) in (12, 13); that is, the scaled wavelet in the frequency space is multiplied by a factor

(14) \[ \sqrt{a} = p^{-\frac{j}{2}} \]
whereas in the "intensity analysis" there is no such multiplicative factor, which means that the higher frequency components are enhanced by a factor of $1/\sqrt{a}$ as compared to the wavelet transformation (12). On the other hand the width of the wavelet in (13) scales for the Morlet as $1/a^2$ and for the intensity analysis method as $1/a$ (narrower shape). In all there are some differences in the equalizing of these methods. Clearly one could develop, in case they turn out to be useful, analyzing methods where the equalizing parameters can be easily set by the user.

References
